Last time: $L^p(X)$ is a Banach space

Today: Hilbert space and $L^2(X)$

Fourier transforms in general

Fix a measure space $(X, \mathcal{M}, \mu)$

Given $\psi_1, \psi_2 \in L^2(X)$ we define

$$\langle \psi_1, \psi_2 \rangle = \int_X \psi_1^* \psi_2 \, d\mu$$

$$= \int_X \psi_1^*(x) \, \psi_2(x) \, d\mu(x)$$

By the Cauchy-Schwarz inequality

(a special case of Hölder's for $p=q=2$)

$$|\langle \psi_1, \psi_2 \rangle| \leq \int |\psi_1^* \psi_2| \, d\mu$$

$$= \| \psi_1^* \psi_2 \|_1$$
$$\leq \| \psi_1^* \|_2 \; \| \psi_2 \|_2$$
$$= \| \psi_1 \|_2 \cdot \| \psi_2 \|_2 < \infty$$

So $\langle \psi_1, \psi_2 \rangle$ is a well-defined complex number.

This defines an inner product on $L^2(X)$ such that $\sqrt{\langle \psi | \psi \rangle} = \| \psi \|_2$.

Since $L^2(X)$ is complete, it is a Hilbert space.

In the homework you will show that any bounded operator L on a Hilbert space $\mathcal{H}$ has an adjoint L^*, which is the unique bounded linear operator st $\langle L^* v, w \rangle = \langle v, Lw \rangle$

for all $v, w \in \mathcal{H}$.

So we can define normal, self-adjoint, and unitary operators as usual.

To understand the spectral theorem for Hilbert spaces, we need to generalize our notion of "diagonal operator".

We'll say a null set to mean a set $N \in \mathcal{M}$ st $\mu(N) = 0$.

Def: A measurable function $f: X \to \mathbb{C}$ is essentially bounded if $\exists$ a null set N such that f restricted to $X \setminus N$ is a bounded function.

As a result of the problem,

$$\mathcal{L}^\infty(X) = \{ f : X \to \mathbb{C} \mid f \text{ is essentially bounded} \}.$$

If $f \in \mathcal{L}^\infty(X)$ and $\psi \in L^2(X)$

$$\|f\psi\|_2^2 = \int_X |f\psi|^2 \, d\mu$$

We know $\exists$ a null set N st

$$\sup_{x \in X \setminus N} |f(x)| = \|f\|_\infty$$

If we set
$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x \in N \\ f(x) & \text{if } x \in X \setminus N \end{cases}$$

Then f and $\tilde{f}$ agree almost everywhere

so
$$\|f\psi\|_2^2 = \int_X |\tilde{f}\psi|^2 \, d\mu$$

$$\leq \int_X \|f\|_\infty^2 \, |\psi|^2 \, d\mu$$

$$= \|f\|_\infty^2 \, \|\psi\|_2^2$$

Hence $\|f\psi\|_2 \leq \|f\|_\infty \|\psi\|_2$.

Given $f \in \mathcal{L}^\infty(X)$, there is a linear operator L_f on $L^2(X)$ defined by

$$L_f(\psi) = f\psi.$$

By the inequality above, $L_f(\psi)$ is also in $\mathcal{L}^2(X)$.
It also tells us that $\|L_f\| \leq \|f\|_\infty$.
When X is nice (σ-finite), $\|L_f\| = \|f\|_\infty$.

$$L_{f_1+f_2} = L_{f_1} + L_{f_2}$$
$$L_{\lambda f} = \lambda L_f$$
$$L_{f_1 f_2} = L_{f_1} L_{f_2}$$

So this defines an algebra map (a linear map preserving products) from

$$\mathcal{L}^\infty(X) \to B(L^2(X))$$

If f is 0 a.e. then $L_f = 0$ exactly.

Since $f\psi$ is 0 a.e. and therefore

$$f\psi = 0 \text{ in } L^2(X).$$

Because $L^\infty(X)$ is $\mathcal{L}^\infty(X) / \{f = 0 \text{ a.e.}\}$

So we get a well-defined map

$$L^\infty(X) \longrightarrow B(L^2(X)).$$

Def: We say that a bounded operator L on $L^2(X)$ is **diagonal** if $L = L_f$ for some $f \in L^\infty(X)$.

The spectral theorem says that every normal operator on a Hilbert space can be made diagonal after a unitary transformation

Recall a unitary map $V \to W$ is a linear isomorphism which preserves inner products.

Thm (Spectral Theorem): Let $\mathcal{H}$ be a Hilbert space and let $L \in B(\mathcal{H})$ be normal. Then there exists a measure space X and a unitary map

$$U : \mathcal{H} \to L^2(X)$$

such that ULU^{-1} is a diagonal operator on $L^2(X)$. That is, $\exists f \in L^\infty(X)$ s.t. $ULU^{-1} = L_f$.

There other ways to state the spectral theorem using projection-valued measure (equivalently, subspace valued measures from the class on d-maps)

$\mathcal{M} \to B(L^2(X))$

e.g. the function $E \mapsto \Pi_E^X$ in the Problem.

measurable set

See the scratch notes in the Drive or Hall's book for more.

We'll just say for now that given a normal operator L on $\mathcal{H}$ there is a canonical spectral decomposition

$$\mathcal{H} \cong \int_{\sigma(L)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$$

direct integral of Hilbert spaces

where $\sigma(L)$ is the spectrum of L and μ is some measure on the Borel sets of $\sigma(L)$.

There's also a version for commuting sets of operators:

Thm: Let S be a set of commuting normal operators on $\mathcal{H}$.
Then we can unitarily diagonalize every element of S simultaneously, i.e. $\exists$ a unitary $U: \mathcal{H} \to L^2(X)$ st $L \mapsto ULU^{-1}$ sends elements of S to diagonal operators.
If S is maximal among sets of commuting normal operators, then this gives a bijection of S with diagonal operators on $L^2(X)$. Furthermore, in this case, X and U are unique up to equivalence.

Let G be a locally compact abelian topological group. That is, G is an abelian group, and a Hausdorff locally compact topological space, such that addition is a continuous map $G \times G \to G$ and inversion is continuous $G \to G$.

E.g. $\mathbb{Z}/N$ or $\mathbb{Z}$ w/ discrete topology

$\mathbb{R}$

$U(1) = S^1 = \mathbb{R}/2\pi\mathbb{Z}$

Such a group has a unique (up to scalar) translation-invariant measure on its Borel sets, satisfying a regularity condition (Radon measure). This is the Haar measure.

E.g. Counting measure on $\mathbb{Z}/n$ and $\mathbb{Z}$

Lebesgue measure on $\mathbb{R}$

Lebesgue masure on S^1

(identifying it with $[0, 2\pi)$)

$\Rightarrow$ Canonical $L^2(G)$ for any such group.

For each $g \in G$, get a unitary operator

$$T_g : L^2(G) \to L^2(G)$$
$$\psi(x) \mapsto \psi(x+g)$$

Since G is abelian, these operators all commute

Let $S = \mathrm{Span}\{T_g \mid g \in G\} \subseteq B(L^2(G))$

It turns out this is a maximal set of commuting normal operators.

By the Spectral Theorem, there is a canonical

$$\mathcal{U} : L^2(G) \xrightarrow{\sim} L^2(X)$$

mapping S bijectively onto $L^\infty(X) \subseteq B(L^2(X))$

the diagonal operators.

It turns out we can take

$$X = \widehat{G} = \{ \chi : G \to U(1) \mid \chi \text{ is a continuous group hom.} \}$$

and $U T_g U^{-1}$ sends $\tilde{\Psi} \in L^2(\hat{G})$

to the function $\chi \mapsto \chi(g) \tilde{\Psi}(\chi)$

The map U sends $\Psi \in L^2(G)$ to its Fourier transform $\tilde{\Psi} = U\Psi \in L^2(\hat{G})$

Exercise: Determine $\hat{\mathbb{Z}}$ and $\hat{\mathbb{R}}$.

If $\Psi \in L^2(G) \cap L^1(G)$ then we can compute

$$\tilde{\Psi}(\chi) = \int_G \overline{\chi(g)}\, \Psi(g)\, d\mu(g).$$